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OPTIMIZATION METHODS FOR ARRAYS OF NONPARALLEL WIRE ANTENNAS.(U)

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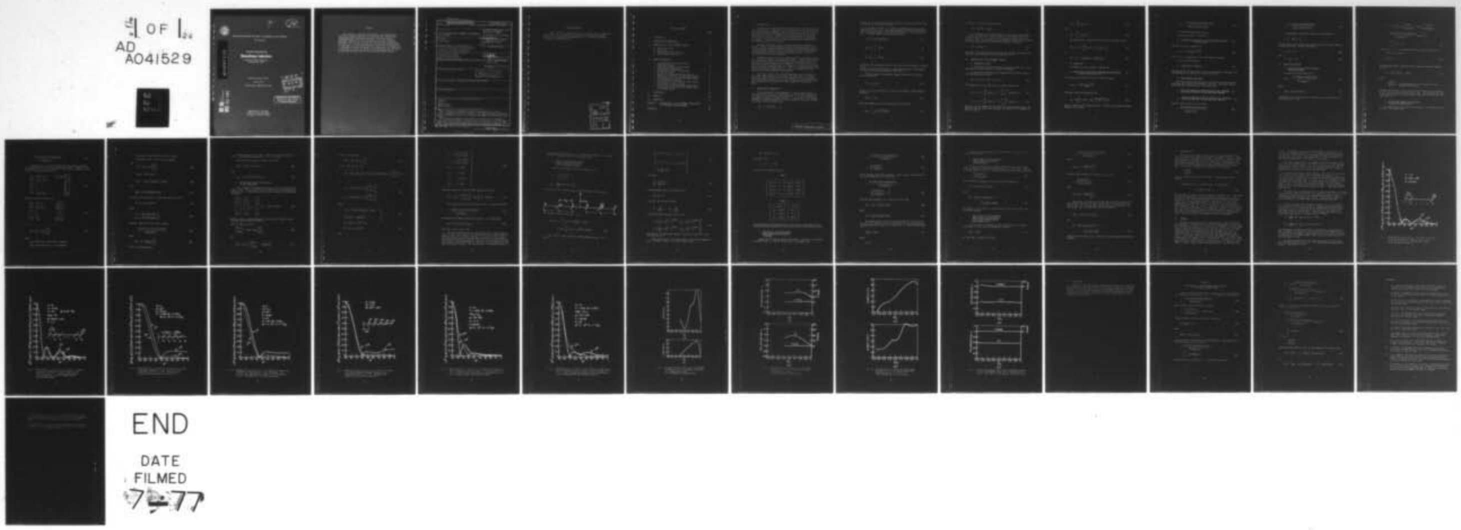

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OPTIMIZATION METHODS FOR ARRAYS OF NONPARALLEL WIRE ANTENNAS

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Matrix methods are presented for optimization problems of arrays of straight, nonparallel, thin wire antennas. The general problem of optimizing certain performance indices with or without any number of constraints can be handled.

The current distribution of any element of the array can be computed as a function of the voltages of the feed points of the array. First the moment method will be briefly given and then it will be shown how the matrices evaluated in the course of this method can be used in optimization problems.

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I. INTRODUCTION

Optimization problems have been investigated the past several years by many researchers. Uzkov[1] first found the solution to the directive gain maximization by employing linear transformations. Bloch, et al.[2], Morgan[3] and Uzsoky and Solymar[4] gave some applications for arrays with arbitrary spacings. Tai[5] considered the problem of achieving maximum gain in uniform linear arrays of dipoles and gave some graphs on the subject. The optimization of SNR with constraints on the Q-factor has been obtained by Lo, et al.[6].

Cheng[7] studied the maximum directive gain by the eigenvalue method. Sunzgiri and Butler[8] have used the eigenvalue method to find the solution for the maximum directive gain with constraints on the resulting sidelobes. Matrix methods were applied by Strait and Kuo[9] for constrained optimization of various performance indices of arrays of straight, parallel thin-wire antennas.

Sahalos[10] gave a solution for maximization of the directive gain of arrays with arbitrarily oriented short dipoles. Sarkar and Strait[14] gave some useful formulas for optimizing certain performance indices of arbitrarily oriented arrays of wire antennas. All these formulas were found assuming known polarization in the specific direction for which the maximum of index is desired, and their study is for optimization problems with only one kind of constraint (i.e., nulls or sidelobes or other indices but not more than one simultaneously).

This paper presents some unified formulations for optimizing any performance index subject to constraints on other indices, sidelobe levels and pattern nulls of arrays with nonparallel thin wire antennas. The formulations presented here are not restricted to optimizations with only one constraint at a time nor do they assume the polarization is known.

II. MOMENT METHOD FORMULATION

The moment method in electromagnetics is very useful because it transforms problems which are formulated in terms of integral equations into systems of simultaneous linear equations. Let a wire structure be composed of a number of straight segments. If we define a right handed orthogonal coordinate system $(n,\phi,\hat{\ell})$ at each point of the wire's cylindrical surface, we can find [11] that:

$$- \int_0^L I(\ell)\,(E_\ell^m - Z_S H_\phi^m)\,d\ell = V_m$$

Equation (1) is the reaction integral equation developed by Richmond and is true for electrically thin wires.

$I(\ell)$ is the total current, ℓ is a metric coordinate measuring position along the wire axis, Z_S is the surface impedance for exterior excitation and L is the wire length. The only unknown in Eq. (1) is the current $I(\ell)$. The other quantities are given as follows:

$$V_m = \iiint (\vec{J}_i \cdot \vec{E}^m - \vec{M}_i \cdot \vec{H}^m) dV \qquad (2)$$

$$E_\ell^m = \frac{1}{2\pi} \int_0^{2\pi} \hat{\ell} \cdot \vec{E}^m \, d\phi \qquad (3)$$

$$E_\phi^m = \frac{1}{2\pi} \int_0^{2} \hat{\phi} \cdot \vec{H}^m \, d\phi \qquad (4)$$

$(\vec{J}_i, \vec{M}_i)$ are the impressed currents by which the wire structure is excited.

$(\vec{E}^m, \vec{H}^m)$ is the field of an electric test source located in the interior of the wire surface and radiating in free space.

The next step is to expand the current on the wire in a finite series of the form:

$$I(\ell) = \sum_n I_n F_n(\ell) \quad . \qquad (5)$$

By Eqs. (1), (5) we can write a system of simultaneous linear algebraic equations

$$[Z] \; (I) = (V) \qquad (6)$$

where the elements Z_{mn} of the matrix [Z] are of the form

$$Z_{mn} = - \int_n F_n(\ell)(E_\ell^m - Z_S H_\phi^m) d\ell \quad . \qquad (7)$$

Solving Eq. (6) for the current we have

$$(I) = [Z]^{-1}(V) = [Y](V) \quad . \tag{8}$$

In an antenna array with N elements, we assume that each element is fed at only one port. From N_1 elements of the column vector (V), ($N_1 = N \cdot N_2$, where N_2 is the number of the expansion modes), only N are non-zero. Thus, we can find a matrix equation which relates the currents on the main ports with the corresponding voltages of the form:

$$(I') = [Y^r](V') \quad . \tag{9}$$

The matrix $[Y^r]$ contains the first N rows and columns of the matrix [Y] which are associated with the non-zero entries in (V).

III. FORMULATION OF THE PERFORMANCE INDICES

A. Antenna Far Field

In finding the solution of an EM optimization problem, it is first necessary to define the total field that is involved.

In the present situation we are interested in the total electric field radiated by an antenna.

$$\vec{E}(r,\phi,\theta) = E_\theta(r,\phi,\theta)\hat{\theta} + E_\phi(r,\phi,\theta)\hat{\phi} \quad . \tag{10}$$

The components $E_\theta(r,\phi,\theta)$ and $E_\phi(r,\phi,\theta)$ can be expressed as

$$E_\theta(r,\phi,\theta) = \frac{1}{r}\sum_{n=1}^{N1} I_n E_{n\theta}(\phi,\theta) = \frac{1}{r}\sum_{i=1}^{N}\sum_{n=1}^{N1} Y_{ni}V_i E_{n\theta}(\phi,\theta) \tag{11}$$

$$E_\phi(r,\phi,\theta) = \frac{1}{r}\sum_{n=1}^{N1} I_n E_{n\phi}(\phi,\theta) = \frac{1}{r}\sum_{i=1}^{N}\sum_{n=1}^{N1} Y_{ni}V_i E_{n\phi}(\phi,\theta) \tag{12}$$

where Y_{ni} are the elements of the matrix [Y], which is given in Eq. (8). We define two row vectors $[B_1]$ and $[B_2]$, such that the i^{th} component of each is given by

$$B_{1i} = \sum_{n=1}^{N1} Y_{ni} E_{n\theta}(\phi,\theta) \tag{13}$$

and

$$B_{2i} = \sum_{n=1}^{N1} Y_{ni} E_{n\phi}(\phi,\theta) \quad . \tag{14}$$

By the above Eqs. (10)-(14) we can write the far zone field as:

$$\vec{E}(r,\phi,\theta) = \frac{1}{r}\left[\sum_{i=1}^{N} B_{1i}V_i\ \hat{\theta} + \sum_{i=1}^{N} B_{2i}V_i\ \hat{\phi}\right] \tag{15}$$

or

$$\vec{E}(r,\phi,\theta) = \frac{1}{r}\left[[B_1][V']\hat{\theta} + [B_2][V']\hat{\phi}\right] \quad . \tag{16}$$

B. <u>Power Gain</u>

The power gain of an antenna is defined by

$$G = \frac{4\pi(\text{radiation intensity for the specified direction})}{\text{Power input to the array}} \quad . \tag{17}$$

In relation to the electric field (17) can be written

$$G(\phi,\theta) = \frac{|\vec{E}(r,\phi,\theta)|^2\ r^2/30}{P_{in}} \quad . \tag{18}$$

The power input can be expressed as:

$$P_{in} = \mathrm{Re}\left(\sum_{i=1}^{N} V_i I_i\right) = [\tilde{V}']^* \left[\frac{[Y^r]+[\tilde{Y}^r]^*}{2}\right][V] \tag{19}$$

where $\sim$ signifies the transpose and the * signifies the complex conjugate. Equation (18) becomes

$$G(\phi,\theta) = \frac{[\tilde{V}']^*[[\tilde{B}_1]^*[B_1]+[\tilde{B}_2]^*[B_2]][V']}{15[\tilde{V}']^*[[Y^r]+[\tilde{Y}^r]^*][V]} \quad . \tag{20}$$

C. Directive Gain and Directivity

The directive gain is defined by:

$$D = \frac{4\pi(\text{radiation intensity for the specified direction})}{\text{radiation power}} \quad . \tag{21}$$

Equation (21) may be expressed as

$$D(\phi,\theta) = \frac{|\vec{E}(r,\phi,\theta)|^2 \cdot r^2/30}{P_r} \tag{22}$$

and $P_r = P_{in} - P_D$, where P_D is the total power dissipated.

It can be shown that[12]

$$P_r = \mathrm{Re}[[\tilde{V}']^*[[Y^r]-[\overline{Y}]][V']] \quad . \tag{23}$$

Complete details for the elements of $[\overline{Y}]$ are available in Reference [12]. The maximum value of the $D(\phi,\theta)$ is the directivity.

D. Other Indices and Factors

Efficiency indices and the sensitivity factors are very important parameters describing the performance of an array. Efficiency indices can be defined in several ways. Two of these are:

$$S = \frac{\text{radiation intensity in the direction of max. radiation}}{\text{sum of the excitation voltage magnitudes squared}} \tag{24}$$

$$S_1 = \frac{\text{radiation intensity in the direction of max. radiation}}{\text{sum of the feed port current magnitudes squared}} \tag{25}$$

Equations (24) and (25) can be written:

$$S = \frac{[\tilde{V}']^*[[\tilde{B}_1]^*[B_1]+[\tilde{B}_2]^*[B_2]][V']}{120\pi[\tilde{V}']^*[V']} \tag{26}$$

$$S_1 = \frac{[\tilde{V}']^*[[\tilde{B}_1]^*[B_1]+[\tilde{B}_2]^*[B_2][V']}{120\pi[\tilde{V}']^*[\tilde{y}^r]^*[Y^r][V']} \quad . \tag{27}$$

Correspondingly, sensitivity factors can be defined as:

$$k = \frac{1}{S} \quad \text{and} \quad k_1 = \frac{1}{S_1} \quad .$$

Another index is the Q-factor that relates gain to efficiency index or sensitivity factor. These are defined as:

$$Q = G/S = G \cdot K \tag{28}$$

and

$$Q_1 = G/S_1 = G\ K_1 \quad . \tag{29}$$

IV. ARRAY OPTIMIZATION

A. Gain-Directive Gain and Efficiency Index Maximization

As we have seen the power gain is

$$G(\phi,\theta) = \frac{[\tilde{V}']^*[[\tilde{B}_1^*][B_1]+[\tilde{B}_2^*][B_2]][V']}{[\tilde{V}']^*[M_2][V']} \tag{30}$$

where

$$[M_2] = 15\ [[Y^r]+[\tilde{Y}^r]^*] \quad . \tag{31}$$

Taking the first variable of G,δG, equal to zero we can find the [V'] which maximize G. Thus,

$$\delta G = [\delta V']^* \left\{ [\tilde{B}_1]^* \frac{[B_1][V']}{[\tilde{V}']^*[M_2][V']} + [\tilde{B}_2]^* \frac{[B_2][V']}{[\tilde{V}']^*[M_2][V']} - \right.$$

$$\left. - \frac{[\tilde{V}']^*[[\tilde{B}_1]^*[B_1]+[\tilde{B}_2]^*[B_2]][V]}{|[\tilde{V}']^*[M_2][V']|^2} [M_2][V] \right\} +$$

$$+ \widetilde{\left\{ \quad \right\}}^* [\delta V'] = 0 \qquad . \tag{32}$$

The quantity $\widetilde{\{\ \}}^*$ is the conjugate transpose vector of the first $\{\ \}$.

By Eq. (32) we have

$$\mathrm{Re} \left\{ \quad \right\} = 0 \tag{33}$$

and so because we are interested only in relative values of voltages, we have:

$$[V'] = [M_2]^{-1}[[\tilde{B}_1]^* + \lambda[\tilde{B}_2]^*] \tag{34}$$

where

$\lambda = \dfrac{[B_2][V']}{[B_1][V']}$ = a value dependent on the polarization in the direction of radiation.

With the same procedure we can find the feed voltages for maximization of the directive gain or any efficiency index. The difference will be in the matrix $[M_2]$, which for the above indices we can find in Eqs. (23), (26), and (27).

B. Optimization Subject to Constraints on Radiation Pattern Nulls

Any index such as for gain, directive gain, or efficiency can be written in the general form:

$$\gamma = \frac{[\tilde{V}']^*[[\tilde{B}_1]^*[B_1]+[\tilde{B}_2]^*[B_2]][V']}{[\tilde{V}']^*[M_2][V']} \quad . \tag{35}$$

Suppose that it is desired to determine feed voltages that will provide pattern nulls in (P-1) directions and maximum γ subject to this constraint in the p^{th} given direction. If Eq. (16) is written for each of the P directions we will have:

$$\begin{bmatrix} \vec{E}_o \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = \begin{bmatrix} (f^1_{11}\hat{\theta}+f^2_{11}\hat{\phi}) & \cdots & (f^1_{1N}\hat{\theta}+f^2_{1N}\hat{\phi}) \\ (f^1_{12}\hat{\theta}+f^2_{12}\hat{\phi}) & \cdot & \cdot \\ \cdot & \cdot & \\ \cdot & \cdot & \\ \cdot & & \cdot \\ \cdot & & \\ \cdot & & \\ (f^1_{P1}\hat{\theta}+f^2_{P1}\hat{\phi}) & \cdots & \cdot \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ \cdot \\ \\ \\ V_N \end{bmatrix} \tag{36}$$

Equation (36) may be modified to:

$$\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \\ 0 \end{bmatrix} = \begin{bmatrix} f^1_{11} & f^1_{12} & \cdot & \cdot & \cdot & f^1_{1N} \\ f^2_{11} & f^2_{12} & \cdot & \cdot & \cdot & f^2_{1N} \\ \cdot & & \cdot & & & \\ \cdot & & & \cdot & & \\ & & & & \cdot & \\ \cdot & & & & & \\ f^2_{P1} & & \cdot & & \cdot & f^2_{PN} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ V_N \end{bmatrix} \tag{37}$$

or

$$[0] = \left[B_{21} \;\vdots\; B_{22} \right] \begin{bmatrix} V_1' \\ V_2' \end{bmatrix} \tag{38}$$

where

$[V_1']$ column vector with N-2(P-1) elements

$[V_2']$ column vector with 2(P-1) elements

$[B_{21}]$ matrix with $[N-2(P-1)] \times 2(P-1)$ elements

$[B_{22}]$ matrix with $2(P-1) \times 2(p-1)$ elements

and

$$[\vec{E}_o] = [\vec{B}_{11} \,\vdots\, \vec{B}_{12}]\begin{bmatrix} V_1' \\ V_2' \end{bmatrix} \quad . \tag{39}$$

From Eq. (38) we have:

$$[V_2'] = - [B_{22}]^{-1}[B_{21}][V_1'] = [D][V_1'] \tag{40}$$

then

$$[\vec{E}_o] = [[\vec{B}_{11}]+[\vec{B}_{12}][D]] \; [V_1'] \quad . \tag{41}$$

We easily can see that $\vec{E}_o$ is of the same form as Eq. (16).

$$\vec{E}_o = [[\Gamma_1]\hat{\theta}+[\Gamma_2]\hat{\phi}][V_1'] \tag{42}$$

where:

$$\left.\begin{aligned} [\Gamma_1] &= [[\vec{B}_{11}]+[\vec{B}_{12}][D]]\cdot\hat{\theta} \\ [\Gamma_2] &= [[\vec{B}_{11}]+[\vec{B}_{12}][D]]\cdot\hat{\phi} \end{aligned}\right\} \tag{43}$$

From Eqs. (40) and (42) the index γ becomes

$$\gamma = \frac{[\tilde{V}_1']^*[[\tilde{\Gamma}_1]^*[\Gamma_1]+[\tilde{\Gamma}_2]^*[\Gamma_2]][V_2']}{[\tilde{V}_1']^*[Q][V_1']} \tag{44}$$

where

$$[Q] = [U \,\vdots\, \tilde{D}^*][M_2] \begin{bmatrix} U \\ \hline D \end{bmatrix} \tag{45}$$

([U] is a unitary matrix).

Now the maximization of index γ under constraints of radiation nulls is achieved by maximization of γ, i.e., Eq. (44).

With the same procedure as before we can find:

$$[V_1'] = [Q]^{-1}[[\tilde{\Gamma}_1]^* + \lambda[\tilde{\Gamma}_2]^*] \tag{46}$$

and

$$\gamma_{max} = [\Gamma_1][Q]^{-1}[[\tilde{\Gamma}_1]^* + \lambda[\tilde{\Gamma}_2]^*] \quad . \tag{47}$$

C. <u>Optimization Subject to Constraints on Side-lobe Level</u>

Next, consider the problem of finding $[V']$ that will maximize the index γ of Eq. (35) in a given direction subject to a constraint on the sidelobe level. Suppose that we have p known sidelobes in p directions. That means:

$$\begin{bmatrix} \vec{E}_1 \\ \vec{E}_2 \\ \cdot \\ \cdot \\ \cdot \\ \vec{E}_p \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \cdot \\ \cdot \\ \cdot \\ k_p \end{bmatrix} E_{o\theta}\hat{\theta} + \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \cdot \\ \cdot \\ \cdot \\ \Lambda_p \end{bmatrix} E_{o\phi}\hat{\phi} = [K]E_{o\theta}\hat{\theta} + [\Lambda]E_{o\phi}\hat{\phi} \tag{48}$$

where $E_{o\theta}$ and $E_{o\phi}$ are the components of the electric field $\vec{E}_o$ in the direction that maximizes the index γ.

From Eq. (48) we can take

$$\begin{bmatrix} [K]E_{o\theta} \\ [\Lambda]E_{o\phi} \end{bmatrix} = [B_{21} \;\vdots\; B_{22}] \begin{bmatrix} V_1' \\ V_2' \end{bmatrix} \tag{49}$$

and

$$[V_2'] = [B_{22}]^{-1} \left[\begin{matrix} [K]E_{o\theta} \\ [\Lambda]E_{o\phi} \end{matrix} - [B_{21}][V_1'] \right] \quad . \tag{50}$$

From Eq. (39) we have

$$[\vec{E}_o] = [\vec{B}_{11} \,\vdots\, \vec{B}_{12}] \begin{bmatrix} V_1' \\ V_2' \end{bmatrix} \quad . \tag{51}$$

So Eqs. (50) and (51) give:

$$[\vec{E}_o] = \left[[\vec{B}_{11}]-[\vec{B}_{12}][B_{22}]^{-1}[B_{21}]\right] [V_1']+[\vec{B}_{12}][B_{22}]^{-1} \begin{bmatrix} [K]E_{o\theta} \\ [\Lambda]E_{o\phi} \end{bmatrix} \tag{52}$$

or

$$\left.\begin{aligned} E_{o\theta} &= [D_\theta][V']+[\pi_{12}^{\theta} \;\; \pi_{22}^{\theta}] \begin{bmatrix} [K]E_{o\theta} \\ [\Lambda]E_{o\phi} \end{bmatrix} \\ E_{o\phi} &= [D_\phi][V_1']+[\pi_{12}^{\phi} \;\; \pi_{22}^{\phi}] \begin{bmatrix} [K]E_{o\theta} \\ [\Lambda]E_{o\phi} \end{bmatrix} \end{aligned}\right\} \tag{53}$$

where

$$\left.\begin{aligned} [D_{\theta/\phi}] &= [[\vec{B}_{11}]-[\vec{B}_{12}][B_{22}]^{-1}[B_{21}]] \cdot \hat{\theta}/\hat{\phi} \\ [\pi_{12}^{\theta/\phi} \pi_{22}^{\theta/\phi}] &= [\vec{B}_{12}][B_{22}]^{-1} \cdot \hat{\theta}/\hat{\phi} \end{aligned}\right\} \quad . \tag{54}$$

The system Eq. (53) gives

$$\vec{E}_o = [[\Sigma_1]\hat{\theta}+[\Sigma_2]\hat{\phi}][V'] \tag{55}$$

and

$$\left.\begin{aligned}
[\Sigma_1] &= \frac{a_{22}[D_\theta]-a_{21}[D_\phi]}{a_{22}a_{11}-a_{21}a_{12}} \\
[\Sigma_2] &= \frac{-a_{12}[D_\theta]+a_{11}[D_\phi]}{a_{22}a_{11}-a_{21}a_{12}} \\
a_{11} &= 1 - [\pi^\theta_{12}][K] \\
a_{12} &= \quad - [\pi^\theta_{12}][\Lambda] \\
a_{21} &= \quad - [\pi^\phi_{22}][K] \\
a_{22} &= 1 - [\pi^\phi_{12}][\Lambda]
\end{aligned}\right\} \tag{56}$$

With the help of Eqs. (50) and (53), $[V_2']$ takes the form:

$$[V_2'] = [B_{22}]^{-1}\left[\begin{bmatrix}[K][\Sigma_1]\\ [\Lambda][\Sigma_2]\end{bmatrix} - [B_{21}]\right][V_1'] = [D][V_1'] \quad . \tag{57}$$

The performance index with the help of Eqs. (55) and (57) becomes

$$\gamma = \frac{[\tilde{V}_1']^*[[\tilde{\Sigma}_1]^*[\Sigma_1]+[\tilde{\Sigma}_2]^*[\Sigma_2]][V']}{[\tilde{V}_1']^*[Q][V_1']} \tag{58}$$

and from the procedure as before, the maximum γ is obtained when

$$[V_1'] = [Q]^{-1}[[\tilde{\Sigma}_1]^*+\lambda[\tilde{\Sigma}_2]^*] \tag{59}$$

where [Q] is given by Eq. (45).

The above procedure gives only the maximum index γ when we know p values of the electric field. For the sidelobe levels to be the desired values we must use an iterative procedure. So, we start from a known radiation pattern which has the desired sidelobes and use the values of this pattern in the above procedure. With this we can find feed voltages $[V_1']$ and the level and directions of sidelobes. These sidelobes are compared with desired results. The new directions of sidelobes are used to

find new feed voltages $[V_j^i]$, etc. The iterative procedure is continued until all sidelobes will be of the desired values.

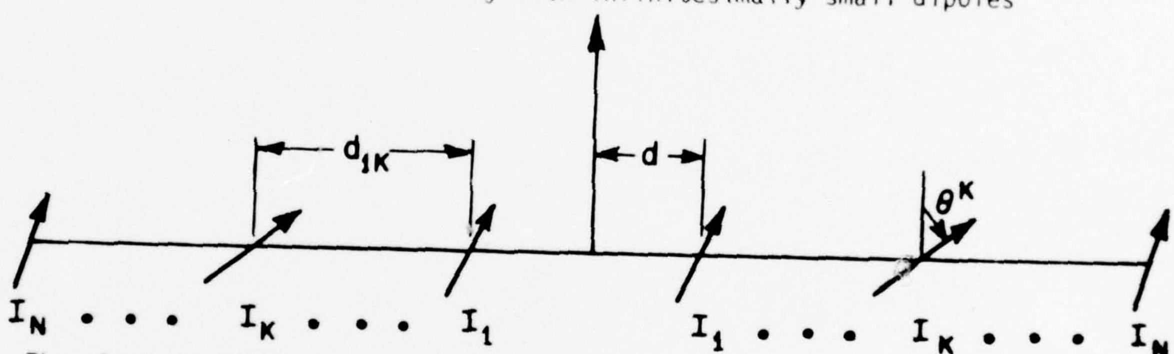
D. Example of a Radiation Pattern Useful as the Starting Point for the Iterative Procedure

Suppose that we want to find an antenna which has

$$\text{(i)} \quad \frac{|E^{\theta}(\pi/2,\pi/2)|}{|E^{\phi}(\pi/2,\pi/2)|} = k$$

$$\text{(ii)} \quad \frac{\text{Main}}{\text{Side}} \text{ lobe ratio} \geq \frac{R^2}{4} \quad .$$

We design a linear uniform array with infinitesimally small dipoles



The electric field on the plane $\theta=\pi/2$ is:

$$E(\phi,\pi/2) = 2 \sum_{i=1}^{N} I_i \cos\theta^i \cos\left(\frac{2i-1}{2}\; d\cos\phi\right) \cdot \hat{\theta} +$$

$$+ 2 \sin\phi \sum_{i=1}^{N} I_i \sin\theta^i \cos\left(\frac{2i-1}{2}\; d\cos\phi\right) \hat{\phi} \quad . \qquad (60)$$

This electric field is that of a Dolph-Chebyshev array. So if we put

$$\left.\begin{aligned}
&\sum_{i=1}^{N} I_i \cos\theta^i \, T_{2i-1}(x) = T_{2N-1}(ax) \\
&\sum_{i=1}^{N} I_i \sin\theta^i \, T_{2i-1}(x) = T_{2N-1}(bx) \\
&x = \left(\cos\left(\frac{d}{2}\cos\phi\right)\right)
\end{aligned}\right\} \tag{61}$$

we obtain

$$\frac{R_a}{R_b} = \frac{T_{2N-1}(a)}{T_{2N-1}(b)} = k \quad . \tag{62}$$

For the maximum value of sidelobe to be 2,

$$R_a^2 + R_b^2 = R^2 \quad . \tag{63}$$

From Eqs. (62) and (63) we have:

$$R_a = \frac{R}{\sqrt{k^2+1}} \quad , \quad R_b = \frac{kR}{\sqrt{k^2+1}} \quad . \tag{64}$$

From R_a and R_b we can define a and b as:

$$\left.\begin{aligned}
a &= \frac{1}{2}\left[\left(R_a + \sqrt{R_a^2 - 1}\right)^{1/2N-1} + \left(R_a - \sqrt{R_a^2 - 1}\right)^{1/2N-1}\right] \\
b &= \frac{1}{2}\left[\left(R_b + \sqrt{R_b^2 - 1}\right)^{1/2N-1} + \left(R_b - \sqrt{R_b^2 - 1}\right)^{1/2N-1}\right]
\end{aligned}\right\} \quad . \tag{65}$$

We can find, from the Dolph procedure, the $I_i\cos\theta^i$ and $I_i\sin\theta^i$ and naturally the I_i, θ^i.

More analytically, if we want to have an array with 8-dipoles of equal distance $\lambda/2$ and $k = \sqrt{3}$, R=40, from Eq. (60)

$R_a = 36.64$, $R_b = 20$

and by Eq. (65)

$a = 1.2$, $b = 1.15$.

The results are shown in Table I.

TABLE I

$I_1 \sin\theta^1 = 3.1$	$I_1 \cos\theta^1 = 9.85$
$I_2 \sin\theta^2 = 2.6$	$I_2 \cos\theta^2 = 3.84$
$I_3 \sin\theta^3 = 1.7$	$I_3 \cos\theta^3 = 0.595$
$I_4 \sin\theta^4 = 1.0$	$I_4 \cos\theta^4 = 0.28$

TABLE II

$I_1 = 30.6$	$\theta^1 = 5^\circ.8$
$I_2 = 4.64$	$\theta^2 = 34^\circ.1$
$I_3 = 1.80$	$\theta^3 = 70^\circ.7$
$I_4 = 1.04$	$\theta^4 = 74^\circ.4$

An appropriate design now would be to use an array of wires with the side-lobes in the same directions as those of the array of infinitesimal dipoles.

E. Optimization of One Performance Index Subject to a Constraint on Another

Suppose that we want to maximize the index γ_1 subject to a constraint on another index γ_2. The two indices are of the general form

$$\gamma_1 = \frac{[\tilde{V}']^*[[\tilde{B}_1]^*[B_1]+[\tilde{B}_2]^*[B_2]][V']}{[\tilde{V}']^*[M_2][V']} \tag{66}$$

$$\gamma_2 = \frac{[\tilde{V}']^*[M_3][V']}{[\tilde{V}']^*[M_4][V']} \quad . \tag{67}$$

By the Lagrange method for maximizing γ_1 when γ_2 has a given value, we must find voltage [V'] which maximizes

$$L = \frac{[\tilde{V}']^*[[\tilde{B}_1]^*[B_1]+[\tilde{B}_2]^*[B_2]][V']}{[\tilde{V}']^*[M_2][V']} +$$

$$+ \lambda \left[\frac{[\tilde{V}']^*[M_3][V']}{[\tilde{V}']^*[M_4][V']} - \gamma_2 \right] \quad . \tag{68}$$

With the same procedure, as in IV-A, we can find that:

$$[V'] = [K]^{-1} \; [[\tilde{B}_1]^* + \mu[\tilde{B}_2]^*] \tag{69}$$

where

$$[K] = [[M_2]+P[\gamma_2[M_2]-[M_3]]] \quad . \tag{70}$$

The unknown value p we can find with the help of the constraint Eq. (67). By a method similar to that of Winkler and Schwartz[13] the substitution of [V'] into Eq. (67) can give P from the real eigenvalues a of

$$[G][X] = a[X] \tag{71}$$

where

$$a = \frac{1}{P} \quad .$$

One of the real eigenvalues gives the optimum γ_1 subject to the correct constraint γ_2,

F. Optimization of One Performance Index Subject to Constraint on Other Indices

Next, we consider the more general problem of maximization of the index γ_1 subject to constraints on other indices $\gamma_2, \gamma_3, \cdots, \gamma_N$. Index γ_1 is of the same form as before, and the other indices are given as follows:

$$\gamma_i = \frac{[\tilde{V}']^*[M_i][V']}{[\tilde{V}']^*[M_{i+1}][V']} \quad . \tag{72}$$

By the method of Lagrange we can find again that [Appendix I]

$$[V'] = [K]^{-1}[[B_1]^* + \mu[B_2]^*] \tag{73}$$

where:

$$[K] = [M_2] + P_1[\gamma_2[M_4] - [M_3]] + \cdots\cdots$$

$$\cdots\cdots + P_N[\gamma_N[M_{N+1}] - [M_N]] \quad . \tag{74}$$

The values P_i can be found by substituting Eq. (73) into Eq. (72), the constraint equation.

G. Optimization of One Performance Index Subject to Constraint on Other Indices, Pattern Nulls and Sidelobe Levels

As we have seen in previous sections, pattern nulls or sidelobe levels give a linear relation between the voltages

$$[V_2'] = [D][V_1'] \quad . \tag{75}$$

So, the index γ_1 becomes as follows:

$$\gamma_1 = \frac{[\tilde{V}_1']^*[[\tilde{\Gamma}_1]^*[\Gamma_1]+[\Gamma_2]^*[\tilde{\Gamma}_2]][V_1']}{[\tilde{V}_1']^*[Q_2][V_1']} \tag{76}$$

where

$$[Q_2] = \left[U \;\vdots\; \tilde{D}^*\right][M_2]\begin{bmatrix} U \\ --- \\ D \end{bmatrix} \quad . \tag{77}$$

Equation (75) transforms all indices $\gamma_2, \gamma_3, \cdots \gamma_N$ as

$$\gamma_i = \frac{[\tilde{V}_1']^*[Q_i][V_1']}{[\tilde{V}_1']^*[Q_{i+1}][V_1']} \tag{78}$$

where

$$[Q_i] = \left[U \;\vdots\; \tilde{D}^*\right][M_i]\begin{bmatrix} U \\ --- \\ D \end{bmatrix} \quad . \tag{79}$$

Our problem of optimization of one index subject to constraints on other indices, pattern nulls and sidelobe levels is achieved with the help of Eqs. (75), (76) and (78). We can find the vector $[V_1']$ by the help of Eqs. (73) and (74) which will be written as:

$$[V_1'] = [K]^{-1}\,[[\Gamma_1]^*+\mu[\Gamma_2]^*] \tag{80}$$

and

$$[K] = [Q_2]+P_1[\gamma_2[Q_4]-[Q_3]] + \cdots$$

$$\cdots + P_n[\gamma_n[Q_{N+1}]-[Q_N]] \quad . \tag{81}$$

Equations (80) and (81) give a solution to the more general optimization problems.

V. POLARIZATION

One of the main problems in our study is the polarization in the specific direction for which the maximum of one index is desired. Sarkar and Strait[14] assume that this is known. The truth is that only arrays with parallel wire antennas give us a known value of polarization. An array with nonparallel wires has a specific value of polarization which we must find. Equation (80) gives a general formula for the voltages $[V_1']$. The number λ dependent on the polarization is given by:

$$\lambda = \frac{[\Gamma_2][V_1']}{[\Gamma_1][V_1']} \quad . \tag{82}$$

From Eqs. (80) and (82) we can find that λ is the solution of the equation:

$$\lambda^2[\Gamma_1][K]^{-1}[\tilde{\Gamma}_2]^\star + \lambda\{[\Gamma_1][K]^{-1}[\tilde{\Gamma}_1]^\star - [\Gamma_2][K]^{-1}[\tilde{\Gamma}_2]^\star\} +$$
$$+ [\Gamma_2][K] \quad [\tilde{\Gamma}_1]^\star = 0 \quad . \tag{83}$$

When we have only linear relations between the voltages, i.e., constraints on nulls and sidelobes, Eq. (83) gives directly the value of λ. Constraints of other indices give nonlinear relations between the voltages and in this case an iterative method can give us the λ. At first we suppose that λ is known and given by Eq. (83) where the matrix [K] is equal to $[Q_2]$ from Eq. (77). Using this value of λ we can find matrix [K] by which Eq. (83) gives a new λ. The new λ is used to find a new [K], etc. The iterative procedure is continued until the new λ is the same as the previously found one. Numerical results obtained by computer showed that there are two equivalent values of λ which give the same maximum index. Each one of these is the negative inverse of the other.

VI. EXAMPLES

1. Consider the design of 6 and 10-element linear arrays of parallel, half-wavelength, centerfed wires to provide optimum indices in the broadside direction ($\phi_0=\pi/2, \theta_0=\pi/2$). The wires are assumed all of radius 0.001λ. Figure 1 shows the normalized $E^\theta(\phi,\pi/2)$ and $E^\phi(\phi,\pi/2)$ fields of a linear array with 6 elements in equal distance $\lambda/2$ and direction $\phi^1=0, \theta^1=\pi/4$. The maximum gain is G=10.97 subject to the constraint that nulls are required in both the direction ($\phi=66°, \theta=90°$) and the direction ($\phi=48°, \theta=90°$). A 10-element linear array with the same geometry as above has maximum gain 16.00 subject to the constraints that: (i) nulls are required in the directions ($\phi=58.6°, \theta=90°$), ($\phi=34°, \theta=90°$), (ii) the $E^\theta(\pi/2,\phi)$ sidelobe levels are 1/5 the level of the main beam and (iii) the efficiency index S is 48.0. Figure 2 shows the normalized $E^\theta(\phi,\pi/2)$ and $E^\phi(\phi,\pi/2)$

fields. A 6-element linear array with the elements in directions $\phi^i=0$, $\theta^i=(i-1)\pi/10$ and equal distance $\lambda/2$, has optimum broadside polarization P=1.720 or P=-.579 subject to the constraint that a null is required in the direction $\phi=50°,\theta=90°$. The maximum gain is 10.24 and in Fig. 3 we can see the normalized $E^\theta(\phi,\pi/2)$ and $E^\phi(\phi,\pi/2)$ fields.

The same 6 element array is considered subject to constraints that: (i) a null is in the direction $\phi=55°,\theta=90°$, (ii) the $E^\theta(\phi,\pi/2)$ sidelobe levels are less than 1/10 and, (iii) the efficiency index is S=65 with optimum polarization P=1.47 or -0.68, and maximum gain 8.89. Figure 4 shows the $E^\theta(\phi,\pi/2)$ and $E^\phi(\phi,\pi/2)$ normalized fields.

If nulls were required in the directions $\phi=60°,\theta=90°$ for the E^θ and $\phi=40°,\theta=90°$ for the E^ϕ field, then the maximum gain goes to 11.2. In Fig. 5 we can see the fields in $\theta=\pi/2$ plane for a 10-element linear array with the elements in directions $\phi^i=0°$ and $\theta^i=(i-1)\pi/18$ and equal distance $\lambda/2$ subject to constraints that: (i) there are nulls in directions $\phi=75°$, $\theta=90°$ for the E^θ and $\phi=60°,\theta=90°$ for the E^ϕ field, (ii) the sidelobe levels of $E^\theta(\phi,\pi/2)$ field are less than (1/10) and, (iii) the efficiency index is S=71.0 with optimum polarization P=1.542, or -.648, and maximum gain G=20.64.

The same array is considered with (i) nulls in directions $\phi=70°,\theta=90°$ for E^θ and $\phi=35°,\theta=90°$ for E^ϕ field, (ii) gain G=20 and, (iii) $E^\theta(\phi,\pi/2)$, sidelobe levels less than 1/12.5, has optimum polarization P=1.796, or -.557, and maximum efficiency index S=94.38. Figures 6 and 7 show the $E^\theta(\phi,\pi/2)$ and $E^\phi(\phi,\pi/2)$ normalized fields for the above two cases.

2. Directivity versus the interelement distance d/λ for a 10-element linear array is shown in Figs. 8a and b. Elements have the directions $\phi^i=0,\theta^i=(i-1)\pi/18$. In Fig. 8a we can see the maximum directivity under constraints on nulls in directions ($\phi=75°,\theta=90°$), ($\phi=40°,\theta=90°$) for E^θ and ($\phi=60°,\theta=90°$) for the E^ϕ field. Figure 8b shows the maximum directivity under constraints on sidelobe levels as follows:

(i) $\frac{\text{side}}{\text{main}}$ lobe $= \frac{1}{10}$ for the E^θ field.

(ii) $\frac{\text{side}}{\text{main}}$ lobe $= \frac{1}{12.5}$ for the E^ϕ field.

The corresponding optimum polarization for both cases is represented in Figs. 9a and b. The maximum directivity subject to constraints on nulls for two different 10-element arrays we have on Fig. 10. Nulls are assumed in directions ($\phi=75°,\theta=90°$),($\phi=105°,\theta=90°$) for the E^θ field and ($\phi=60°$, $\theta=90°$) for the E^ϕ field.

The element directions for the first array are $\phi^i=0,\theta^i=(i-1)\pi/36$ while for the second are $\phi^i=0,\theta^i=\pi/6$. The corresponding optimum polarization is represented in Fig. 11.

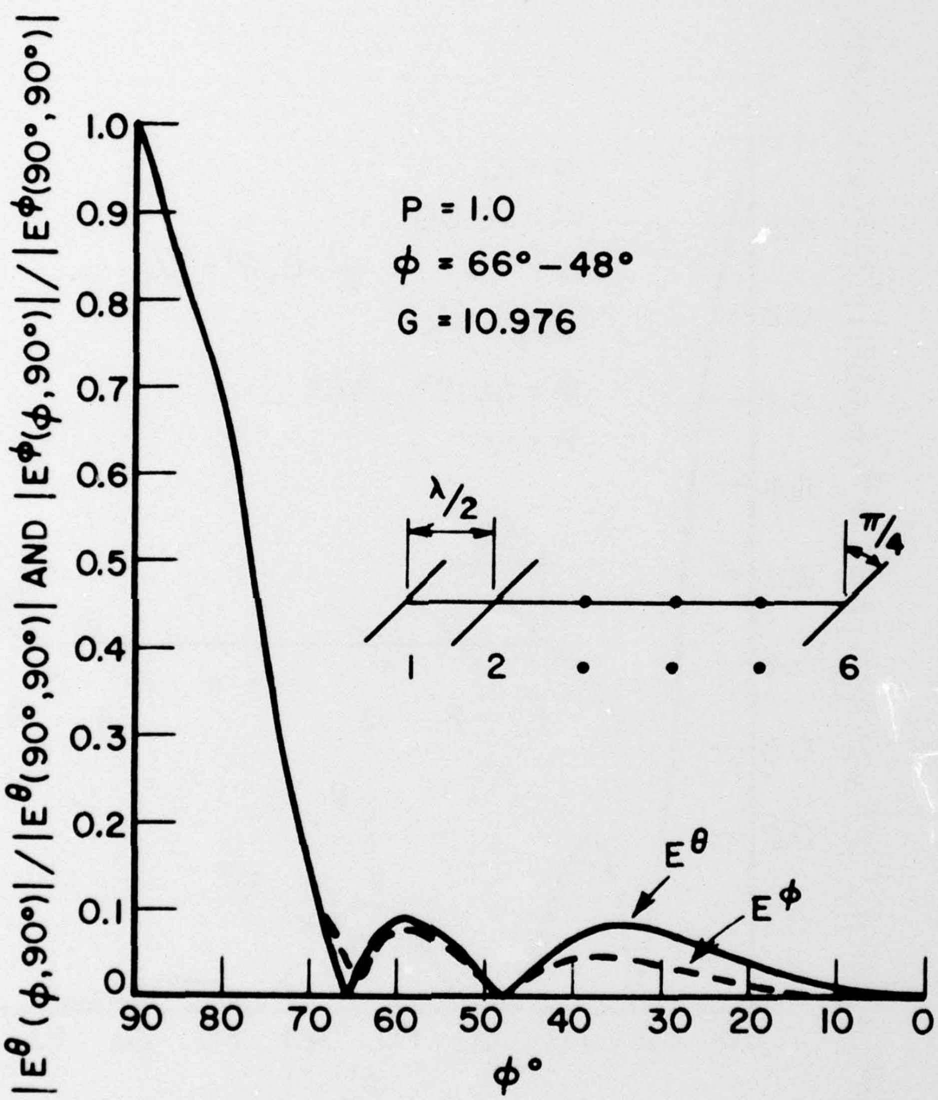


Fig. 1. Normalized $E^{\theta}(\phi,\pi/2)$ and $E^{\phi}(\phi,\pi/2)$ fields of a linear array with 6 elements in equal distance $\lambda/2$ and direction θ^{i}=0, ϕ^{i},$\pi/4$. (Maximum gain subject to the constraint on nulls).

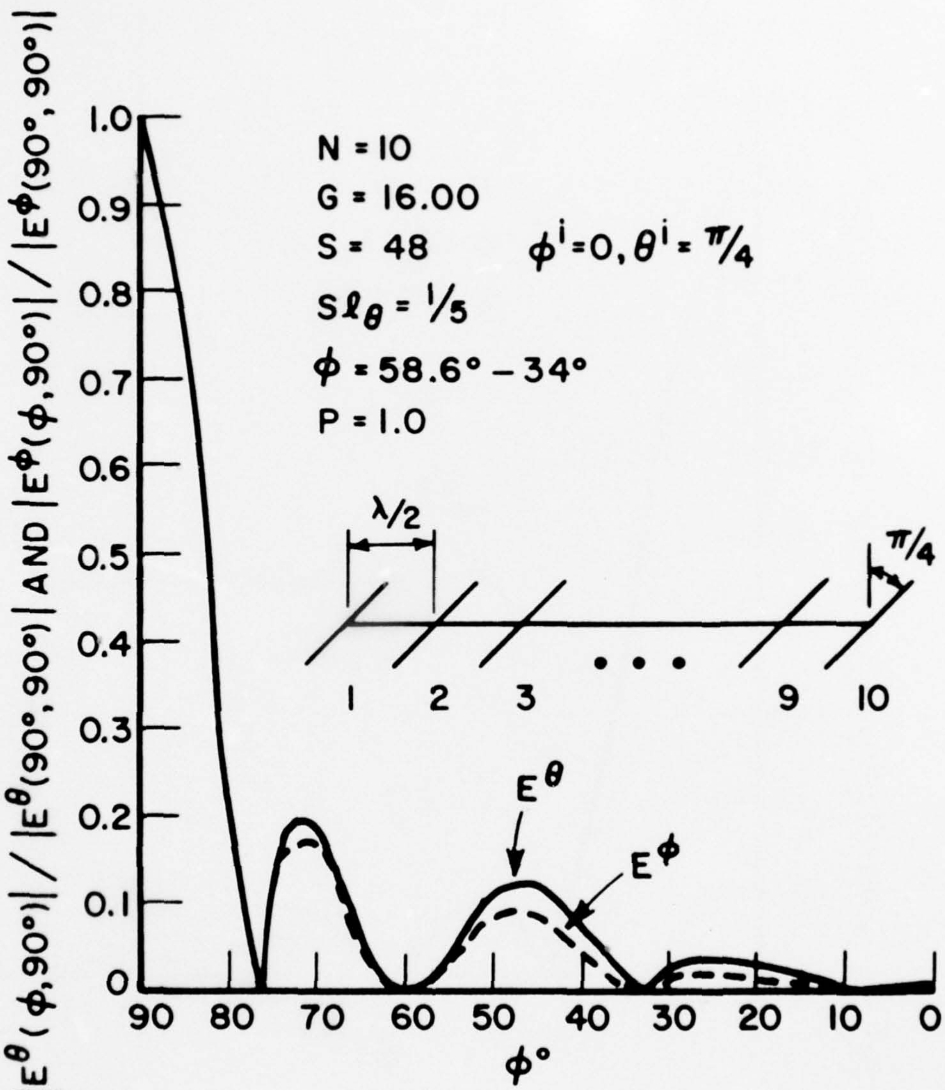


Fig. 2. Normalized $E^{\theta}(\phi,\pi/2)$ and $E^{\phi}(\phi,\pi/2)$ fields of a linear array with 10 elements in equal distance $\lambda/2$ and directions $\theta^{i}=0$, $\phi^{i}= /4$. (Maximum gain subject to the constraints on nulls, sidelobe levels and efficiency index.)

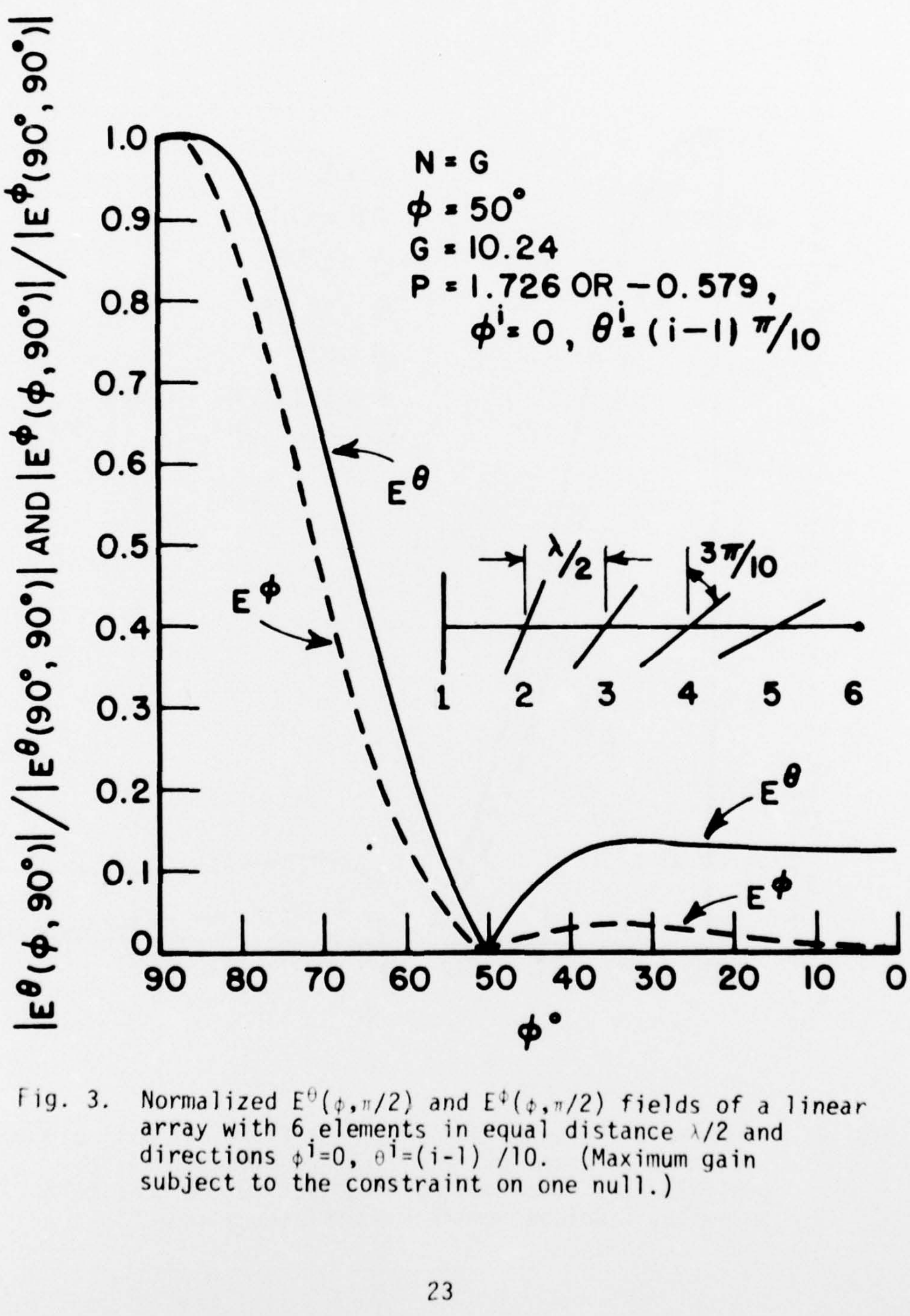


Fig. 3. Normalized $E^{\theta}(\phi,\pi/2)$ and $E^{\phi}(\phi,\pi/2)$ fields of a linear array with 6 elements in equal distance $\lambda/2$ and directions $\phi^i=0$, $\theta^i=(i-1)$ /10. (Maximum gain subject to the constraint on one null.)

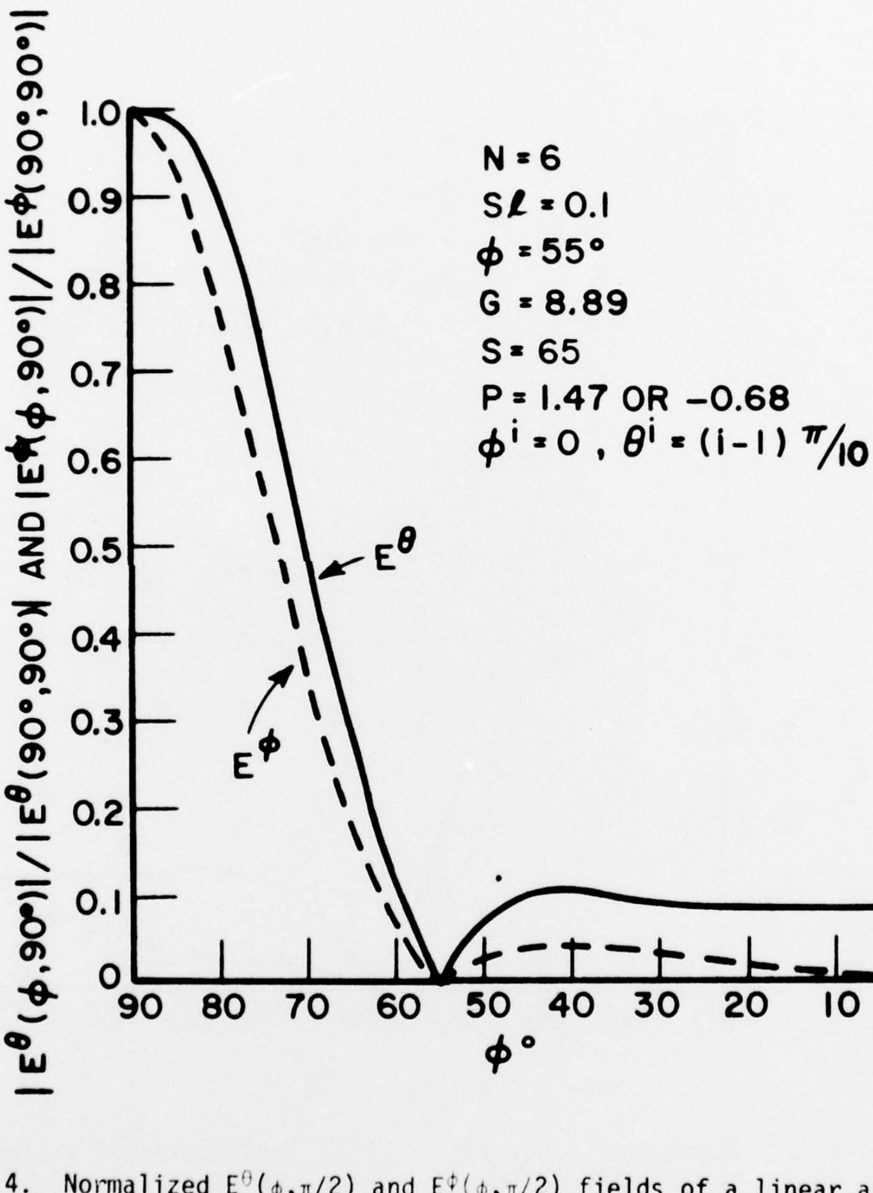


Fig. 4. Normalized $E^{\theta}(\phi,\pi/2)$ and $E^{\phi}(\phi,\pi/2)$ fields of a linear array 6 element in equal distance $\lambda/2$ and directions $\phi^{i}=0$, $\theta^{i}=(i-1)\pi/10$. (Maximum gain subject to the constraints on nulls, sidelobe levels and efficiency index.)

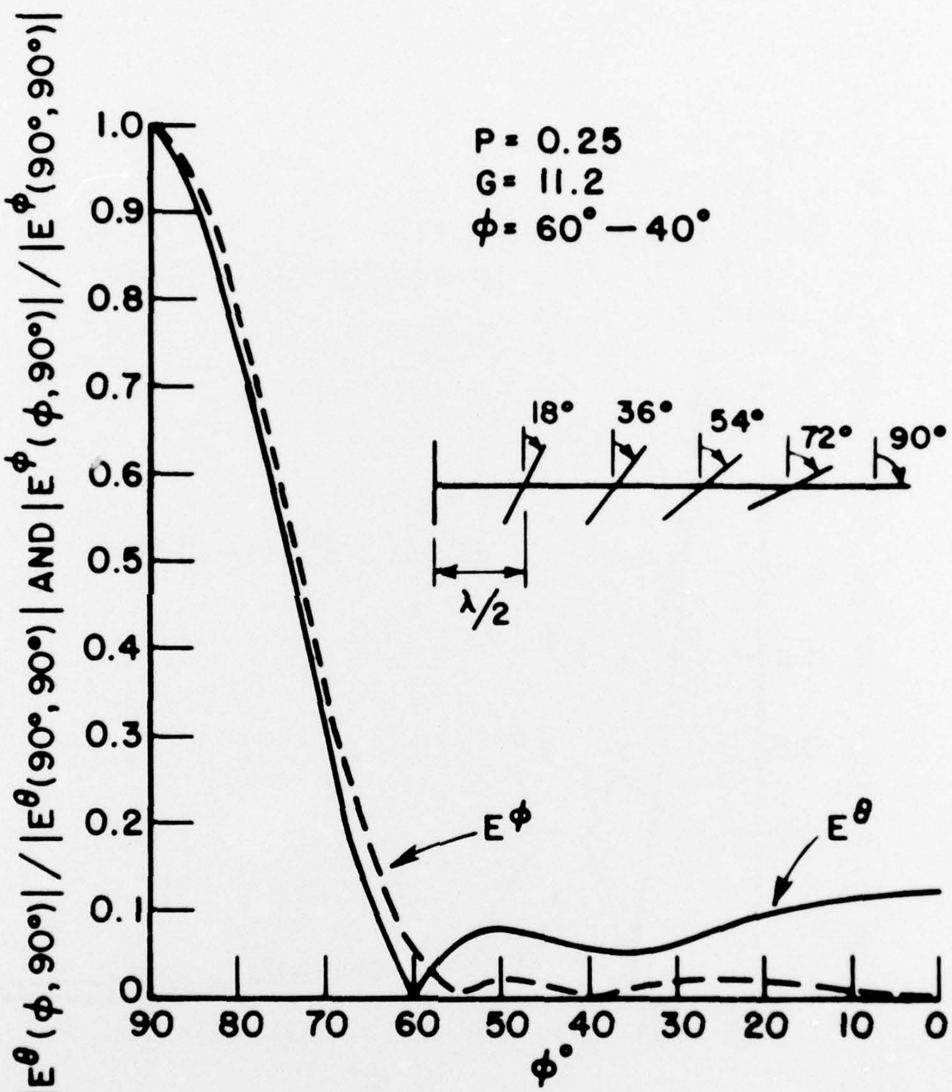


Fig. 5. Normalized $E^\theta(\phi,\pi/2)$ and $E^\phi(\phi,\pi/2)$ fields of a linear array with 6 elements in equal distance $\lambda/2$ and directions $\phi^i=0$, $\theta^i=(i-1)/10$. (Maximum gain subject to the constraint on nulls.)

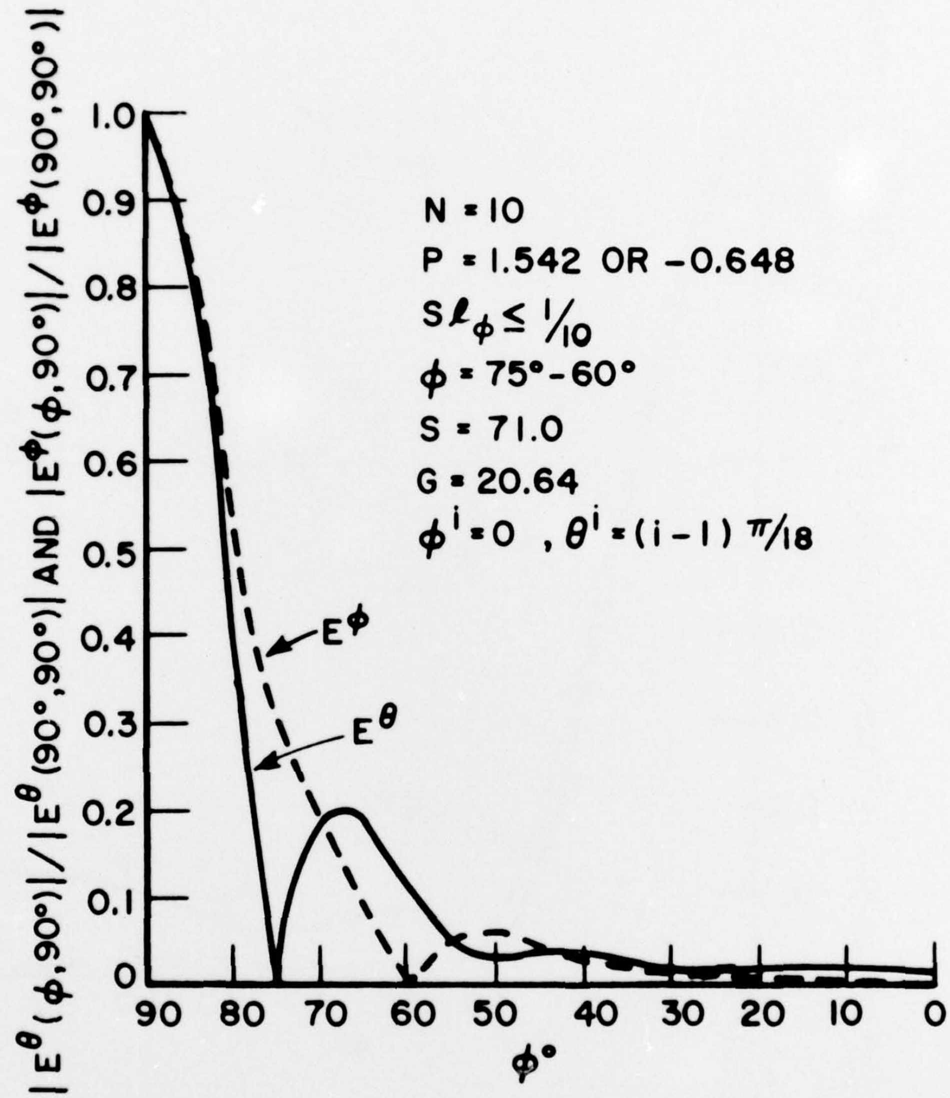


Fig. 6. Normalized $E^\theta(\phi,\pi/2)$ and $E^\phi(\phi,\pi/2)$ fields of a linear array with 10 elements in equal distance $\lambda/4$ and directions $\phi^i=0$, $\theta^i=(i-1)\pi/18$. (Maximum gain subject to the constraints on nulls, sidelobe levels and efficiency index.)

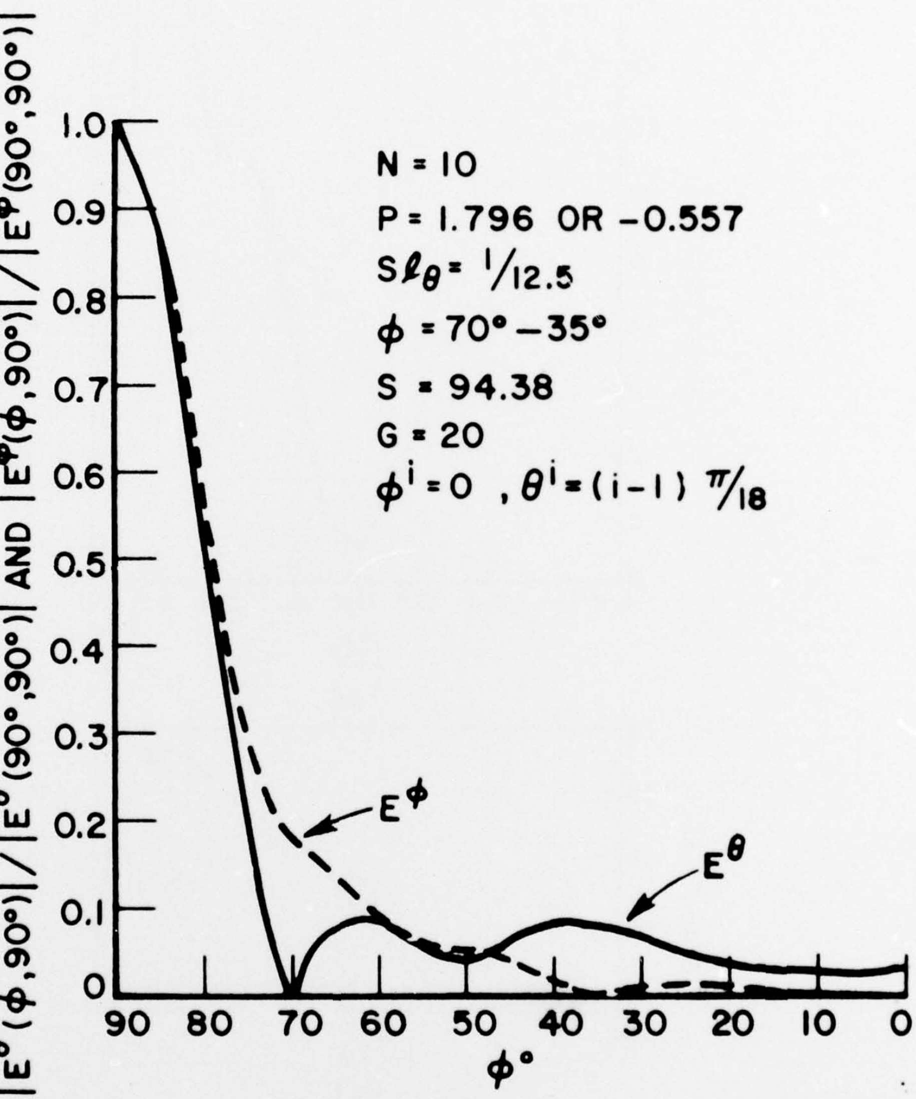


Fig. 7. Normalized $E^\theta(\phi,\pi/2)$ and $E^\phi(\phi,\pi/2)$ fields of a linear array with 10 elements in equal distance $\lambda/4$ and directions $\phi^i=0$, $\theta^i(i-1)\pi/18$. (Maximum gain subject to the constraints on nulls, sidelobe levels and efficiency index.)

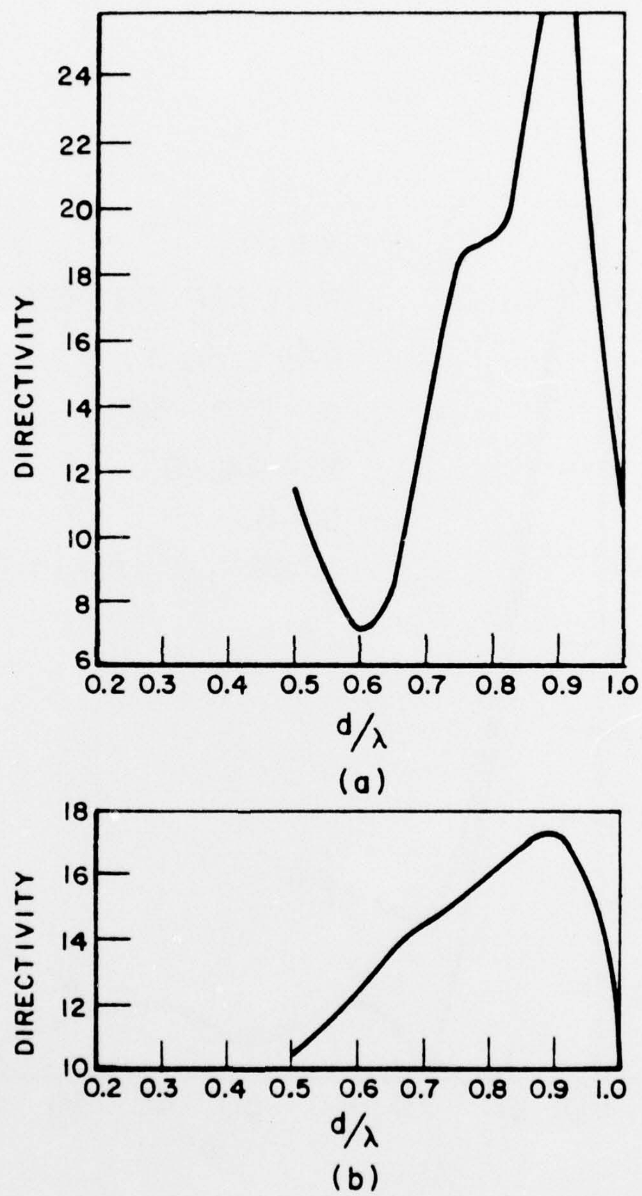


Fig. 8. Maximum directivity versus the interelement distance d/λ for a 10-element linear array
(a) Constraint on nulls
(b) Constraint on sidelobe levels.

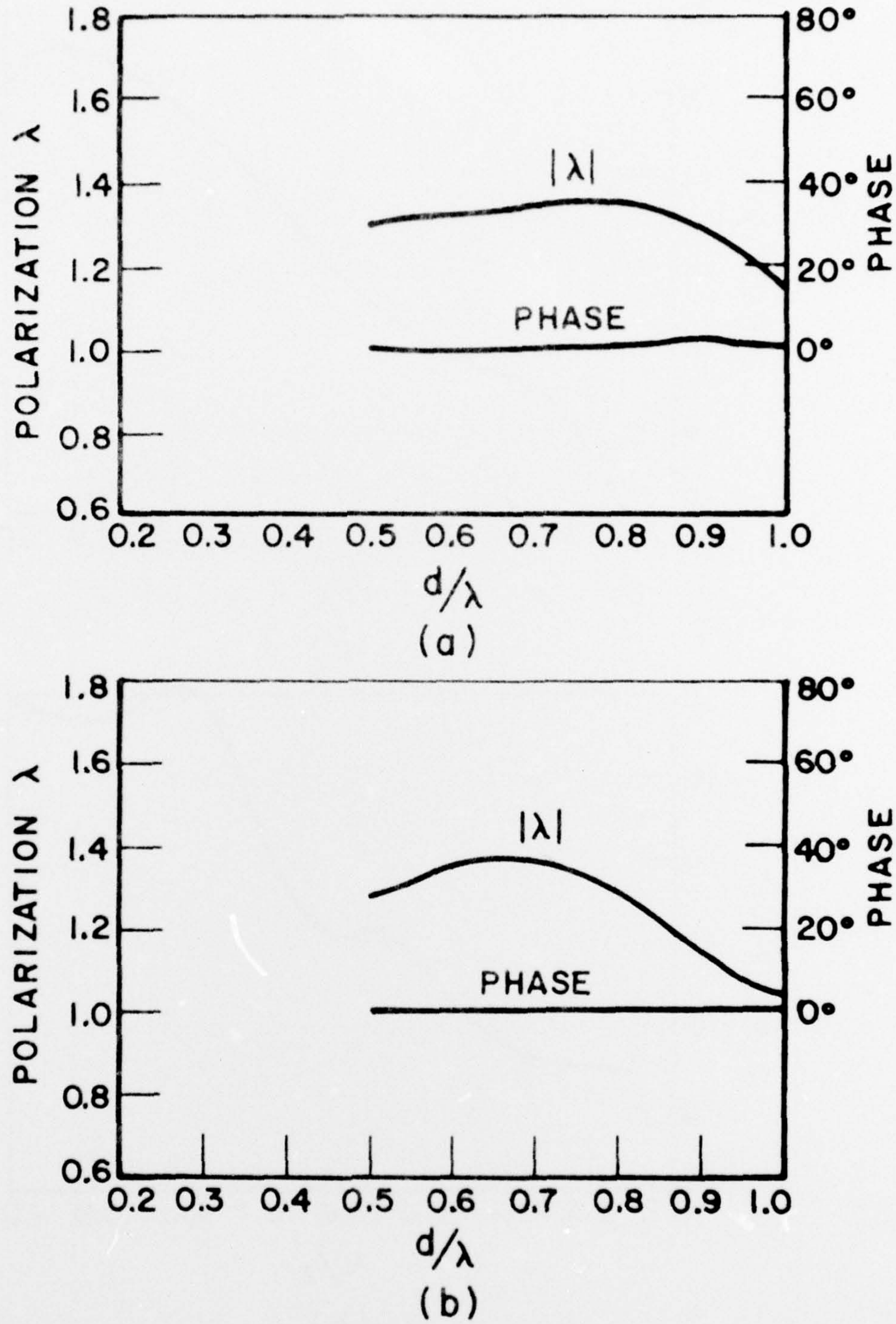
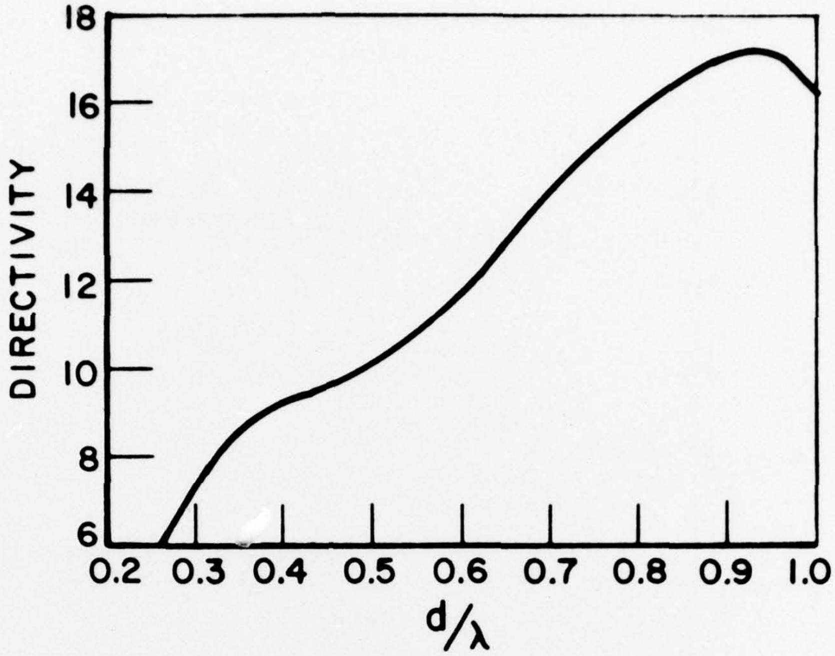
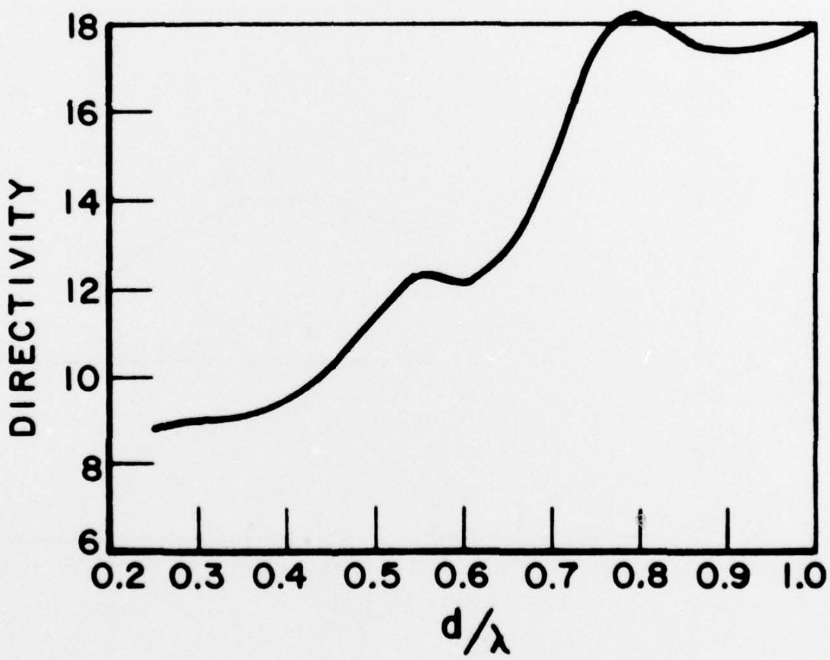


Fig. 9. Optimum polarization versus the interelement distance d/λ for a 10-element linear array
(a) Constraint on nulls
(b) Constraint on sidelobe levels.

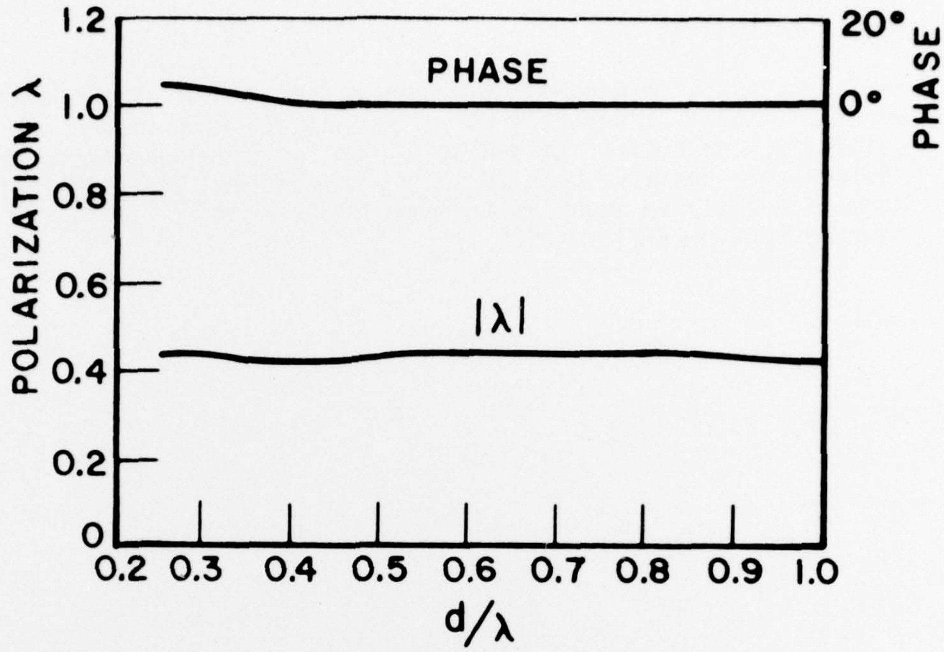


(a)

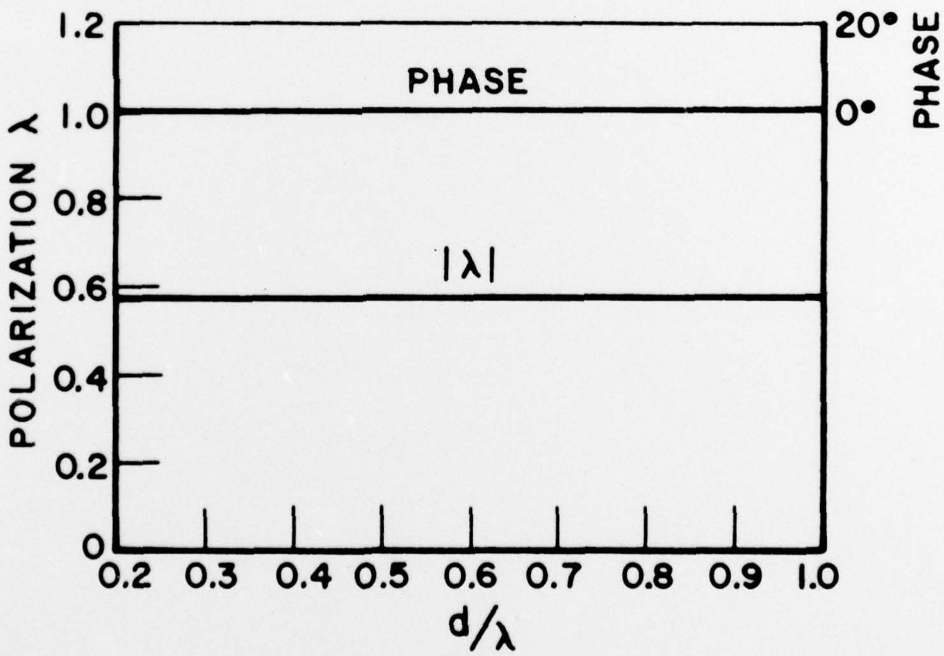


(b)

Fig. 10. Maximum directivity versus the interelement distance d/λ for a 10-element linear array (Constraint on nulls). (a) $\phi^i=0$, $\theta^i=(i-1)\pi/36$, (b) $\phi^i=0$, $\theta^i=(i-1)\pi/6$.



(a)



(b)

Fig. 11. Optimum polarization versus the interelement distance d/λ for a 10-element linear array. (Constraints on nulls) (a) $\phi^{i}=0$, $\theta^{i}=(i-1)\pi/36$, (b) $\phi^{i}=0$, $\theta^{i}=(i-1)\pi/6$.

VII. CONCLUSIONS

Optimization methods have been presented for arrays with nonparallel wire antennas. It has been shown how a performance index can be optimized subject to constraints on the nulls, on the sidelobe levels and on other indices. It has also been shown how we can find the polarization in the direction which an index is maximized. Several examples were included to show some possible uses.

APPENDIX I

OPTIMIZATION OF ONE PERFORMANCE INDEX SUBJECT TO CONSTRAINT ON OTHER INDICES

Suppose that we have the index γ_1 which must be maximized subject to constraints on other indices $\gamma_2, \gamma_3, \cdots, \gamma_N$. Index γ_1 is of the form

$$\gamma_1 = \frac{[\tilde{V}']*[[\tilde{B}_1]*[B_1]+[\tilde{B}_2]*[B_2]][V']}{[\tilde{V}']*[M_2][V']} \tag{84}$$

and the other indices are given as follows:

$$\gamma_i = \frac{[\tilde{V}']*[M_i][V']}{[\tilde{V}']*[M_{i+1}][V']} = \text{a given value} \quad . \tag{85}$$

The conditions Eq. (85) can be written in the form

$$[\tilde{V}'][MM_i][V'] = 0 \tag{86}$$

where

$$[MM_i] = \gamma_i[M_{i+1}]-[M_i] \quad . \tag{87}$$

For maximizing γ_1 we will use the Lagrange method. That means we will find the maximum of the function L.

$$L = \frac{[\tilde{V}']*[[\tilde{B}_1]*[B_1]+[\tilde{B}_2]*[B_2]][V']}{[\tilde{V}']*[M_2][V']} + \\ + \sum_{i=2}^{N} \lambda_i[\tilde{V}'][MM_i][V'] \quad . \tag{88}$$

Taking the first variation of L, δL, equal to zero we have:

$$\delta L = [\delta\tilde{V}]^* \left\{ \frac{[[\tilde{B}_1]^*[B_1]+[\tilde{B}_2]^*[B_2]][V']}{[\tilde{V}']^*[M_2][V]} - \right.$$

$$\frac{[\tilde{V}']^*[[\tilde{B}_1]^*[B_1]+[\tilde{B}_2]^*[B_2]][V]}{|[\tilde{V}']^*[M_2][V']|^2} [M_2][V'] +$$

$$\left. + \sum_{i=2}^{N} \lambda_i [MM_i][V'] \right\} + \{\tilde{\ }\}^* [\delta V'] = 0 \qquad (89)$$

where $\{\tilde{\ }\}^*$ signifies the conjugate transpose of the first vector $\{\ \}$. From Eq. (89) we have:

$$\frac{[[\tilde{B}_1]^*[B_1]+[\tilde{B}_2]^*[B_2]][V']}{[\tilde{V}']^*[M_2][V']} =$$

$$= \frac{[\tilde{V}']^*[[\tilde{B}_1]^*[B_1]+[\tilde{B}_2]^*[B_2]][V']}{|[\tilde{V}']^*[M_2][V']|^2} [M_2][V'] -$$

$$- \sum_{i=2}^{N} \lambda_i [MM_i][V'] \quad . \qquad (90)$$

If we put

$$\mu = \frac{[B_2][V]}{[B_1][V]} \quad ,$$

and we need the relative values of the voltages of Eq. (90) we find:

$$[V'] = \left[[M_2] + \sum_{i=2}^{N} P_i [MM_i] \right]^{-1} [[\tilde{B}_1]^* + \mu [\tilde{B}_2]^*] \qquad (91)$$

or

$$[V'] = \left[[M_2] + P_2[\gamma_2[M_4]-[M_3]] + \cdots \right]^{-1} [[\tilde{B}_1]^* \mu [\tilde{B}_2]^*] \quad . \qquad (92)$$

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